11456. *Proposed by Raymond Mortini, Université Paul Verlaine, Metz, France.* Find

$$\lim_{n\to\infty} n \prod_{m=1}^{n} \Big(1 - \frac{1}{m} + \frac{5}{4m^2}\Big).$$

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The answer is that the given limit of a product is $(\cosh\pi)/\pi = (e^{\pi} + e^{-\pi})/2\pi$.

We give 3 solutions to this problem. The first is how we first solved the problem, while the 2nd and 3rd are more elementary (or "basic") in that they work more directly from first principles, and call upon fewer (or no) extra facts about the Gamma function. Indeed, our second solution uses essentially only the Gauss definition of the Gamma function and the reflection formula, while the 3rd proof avoids the Gamma function altogether, making a direct connection to trigonometric functions (but requires knowledge of the product expansion for $\sin z$).

In all solutions we use L to denote the limit in question, and P_n to denote the partial product occurring in it (so that $L = \lim_{n\to\infty} (n\, P_n)$).

First Solution: We begin by considering the product. We have

$$\begin{aligned} P_n &:= \prod_{m=1}^{n} \left(1 - \frac{1}{m} + \frac{5}{4m^2}\right) \\ &= \prod_{m=1}^{n} \frac{m^2 - m + 5/4}{m^2} \\ &= \frac{\prod_{m=1}^{n} \left((m-1/2)^2 + 1\right)}{(n!)^2} \\ &= \frac{1}{(n!)^2} \prod_{m=1}^{n} (m - 1/2 + i)(m - 1/2 - i). \end{aligned}$$

Now by the basic factorial property of the Gamma function, $\Gamma(z+1) = z\,\Gamma(z)$, we can rewrite the product occurring here in terms of the Gamma function, obtaining

$$P_n = \frac{1}{(n!)^2} \frac{\Gamma(n+1/2+i)\,\Gamma(n+1/2-i)}{\Gamma(1/2+i)\,\Gamma(1/2-i)},$$

and hence (since n!=$\Gamma(n+1)$)

$$\begin{aligned} P_n &= \frac{\Gamma(n+1/2+i)\,\Gamma(n+1/2-i)}{\Gamma(n+1)^2\,\Gamma(1/2+i)\,\Gamma(1/2-i)} \\ &= \frac{1}{|\Gamma(1/2+i)|^2} \left|\frac{\Gamma(n+1/2+i)}{\Gamma(n+1)}\right|^2. \end{aligned}$$

We are now ready to evaluate the stated limit, which is easy given that the Gamma function enjoys the property that $\Gamma(z+a)/\Gamma(z+b)$ goes as z^{a-b} as $Re\, z$ goes to ∞ for a and b fixed complex numbers (see, for example, F. W. J. Olver, Asymptotics and Special Functions, Academic Press, 1974, pp. 118-119, or G. E. Andrews, R. Askey, and R. Roy, Special Functions, Cambridge University Press, Cambridge, 1999, pp. 615–616; in fact, this behavior is enjoyed as $|z| \to \infty$ in any sector $|\arg z| < \pi - \epsilon$, $\epsilon > 0$). That is, the ratio of $\Gamma(z+a)/\Gamma(z+b)$ to z^{a-b} has limit 1 as $Re\, z$ goes to ∞. Thus $\frac{\Gamma(n+1/2+i)}{\Gamma(n+1)}$ goes like $n^{i-1/2}$ for large n, and its modulus squared behaves like n^{-1} as n goes to ∞. The limit in question is therefore

$$L = \lim_{n\to\infty} (n\, P_n) = \frac{1}{|\Gamma(1/2+i)|^2}$$

$$= \frac{1}{\Gamma(1/2+i)\,\Gamma(1/2-i)},$$

and to evaluate the denominator here we recall one more property of the Gamma function, the reflection property. This says that $\Gamma(z)\,\Gamma(1-z) = \pi/(\sin \pi z)$ and here it gives

$$L = \lim_{n\to\infty} (n\,P_n) = \frac{1}{\Gamma(1/2+i)\,\Gamma(1/2-i)} = \sin(\pi(1/2+i))/\pi$$

$$= \frac{1}{\pi}[\sin(\pi/2)\ \cos(i\pi) + \cos(\pi/2)\ \sin(i\pi)] = \frac{1}{\pi}\cosh\pi,$$

the result claimed above.

Second Solution: Here we make a direct connection to the Gamma function via the identity

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\,n^z}{z(z+1)(z+2)\cdots(z+n)},$$

which holds for all complex values of z excepting 0 and the negative integers (in the limit formula n is an integer tending to ∞, and hence $n!$ makes sense there). As before we have

$$P_n := \prod_{m=1}^{n} \Big(1 - \frac{1}{m} + \frac{5}{4m^2}\Big)$$

$$= \prod_{m=1}^{n} \frac{m^2 - m + 5/4}{m^2}$$

$$= \frac{\prod_{m=1}^{n}((m-1/2)^2+1)}{(n!)^2}$$

$$= \frac{1}{(n!)^2}\prod_{m=1}^{n}(m-1/2+i)(m-1/2-i).$$

Write $z = \frac{-1}{2} + i$, with $z + \bar{z} = -1$, and thus

$$nP_n = \frac{1}{n!n^z}\prod_{m=1}^{n}(m+z)\frac{1}{n!n^{\bar{z}}}\prod_{m=1}^{n}(m+\bar{z}).$$

Taking the limit and using the limit formula,

$$\begin{aligned} L &= \lim(n\,P_n) = \frac{1}{z\Gamma(z)\bar{z}\Gamma(\bar{z})} = \frac{1}{\Gamma(z+1)\Gamma(\bar{z}+1)}, \\ &= \frac{1}{\Gamma(1/2+i)\,\Gamma(1/2-i)} \\ &= \frac{\cosh\pi}{\pi} = \frac{e^{\pi}+e^{-\pi}}{2\pi}, \end{aligned}$$

as above (where we have used the reflection formula for the Gamma function, $\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z}$, to effect the final evaluation).

[Historical note: This identity for the Gamma function was known to Euler (in fact, it was by means of this formula that he first defined the Gamma function as an analytic interpolation of the factorial; see P. J. Davis, "Leonhard Euler's Integral: A Historical Profile of the Gamma Function," AMM **66** (1959), 849–869), though he also went on to obtain an integral representation of the Gamma function equivalent to that which is in common usage today. This limit identity was later taken as the definition by Gauss in his treatment of the Gamma function, and is therefore often referred to as Gauss's formula for the Gamma function, or, perhaps better, as Gauss's definition of the Gamma function. On the other hand, Copson refers to it as "Euler's limit formula for $\Gamma(z)$" (see E. T. Copson, An Introduction to the Theory of Functions of a Complex Variable, Oxford University Press, London, 1944, p. 209).]

Third Solution: Here we work to connect the given product directly with the product expansion of the sine function. This product formula states that

$$\begin{aligned}\sin \pi z &= \pi z \prod_{m=1}^{\infty} \left(1 - \frac{z^2}{m^2}\right) \\ &= \pi z \lim_{n\to\infty} \prod_{m=1}^{n} \left(1 - \frac{z^2}{m^2}\right) \\ &= \pi z \lim_{n\to\infty} \left[\prod_{m=-1}^{-n} \left(1 - \frac{z}{m}\right) \prod_{m=1}^{n} \left(1 - \frac{z}{m}\right)\right]\end{aligned}$$

and thus we would like to be able to identify the limit of this problem as a similar limit, or special case of this limit, at least in general terms.

We begin by looking at P_n, trying to separate its two sets of factors into those corresponding to positive and negative choices of the index m in the sine expansion above. We have

$$\begin{aligned}P_n &= \prod_{m=1}^{n} \frac{(m-1/2)^2+1}{m^2} \\ &= \prod_{m=1}^{n} \frac{m-1/2+i}{m} \, \frac{m-1/2-i}{m} \\ &= \prod_{m=1}^{n} \left(1 + \frac{i-1/2}{m}\right)\left(1 - \frac{i+1/2}{m}\right) \\ &= \prod_{m=-1}^{-n} \left(1 - \frac{i-1/2}{m}\right) \prod_{m=1}^{n} \left(1 - \frac{i+1/2}{m}\right).\end{aligned}$$

We are now confronted with the problem that the z's that we'd like to identify to see the sine expansion at this point are different for the two products. But these z values differ by 1, which makes it possible to "correct" for this defect (any integer difference gives a workable subproblem here, in the sense that it would still be possible to rewrite the limit of an appropriate power of n times P_n as a sine times some algebraic factors, but the smaller the integer in absolute value the easier the subproblem). We can work on either of the two products appearing in the last displayed line above. We choose to work on the factors with positive m, and thus we commit to taking $z = i - 1/2$ (that is, the z that we identify from the factors with negative index m). The correction uses the fact that $(m - 1/2 - i)/m = (m - 1 - z)/m = \frac{m-1}{m}\frac{m-1-z}{m-1} = \frac{m-1}{m}(1 - \frac{z}{m-1})$. We separate off the $m = 1$ factor in the second product (writing it as the leading factor in our expression for P_n) and make the compensating adjustments on the rest, yielding

$$P_n = (1/2 - i) \prod_{m=-1}^{-n} \left(1 - \frac{i-1/2}{m}\right) \prod_{m=2}^{n} \left(\frac{m-1}{m}\right) \prod_{m=2}^{n} \left(1 - \frac{i-1/2}{m-1}\right)$$

(note that the last two products begin with index $m = 2$). At this point we see that the middle product "telescopes" to give $1/n$ and we can make an index shift in the last product to help effect a match with the first product. Using $z = i - 1/2$ we have

$$\begin{aligned}P_n &= \frac{-z}{n} \prod_{m=-1}^{-n} \left(1 - \frac{z}{m}\right) \prod_{m=1}^{n-1} \left(1 - \frac{z}{m}\right) \\ &= -\frac{1}{n\pi} \left(1 - \frac{z}{n}\right)^{-1} \left[\pi z \prod_{m=-1}^{-n} \left(1 - \frac{z}{m}\right) \prod_{m=1}^{n} \left(1 - \frac{z}{m}\right)\right],\end{aligned}$$

and hence, using the product expansion for $\sin \pi z$ exhibited above, we have

$$L = \lim_{n\to\infty} (n\, P_n) = -\frac{1}{\pi} \sin \pi z$$

$$= -\frac{1}{\pi}\,\sin\pi(i-1/2)$$
$$= \frac{1}{\pi}\,\cos(i\pi)$$
$$= \frac{\cosh\pi}{\pi},$$

and we are done.

Remark. This 3rd solution worked by way of putting the initial product into the form of the canonical product expansion for the sine function; yet another solution would be to work to put it into the form of the canonical product for the cosine (which is of course only a short step from that for the sine). In taking this approach one finds that the terms for positive and negative m (as broken up in Solution 3 above) are handled more symmetrically than in Solution 3 (and thus one no longer needs to carry out an index shift on half the factors). But to carry this proof through most directly one also needs to call upon the evaluation of the Wallis product,

$$W = \lim_{n\to\infty}\left[\frac{1\cdot 3}{2\cdot 2}\,\frac{3\cdot 5}{4\cdot 4}\cdots\frac{(2n-1)(2n+1)}{(2n)(2n)}\right] = \frac{2}{\pi}.$$

In short, the proof is accomplished via

$$L = \lim_{n\to\infty}\left(n\prod_{m=1}^{n}\frac{(m-1/2)^2+1}{m^2}\right)$$

$$= \lim_{n\to\infty}\left[n\prod_{m=1}^{n}\left(\frac{m-1/2}{m}\right)^2\prod_{m=1}^{n}\frac{(m-1/2)^2+1}{(m-1/2)^2}\right]$$
$$= \lim_{n\to\infty}\left[n\prod_{m=1}^{n}\left(\frac{2m-1}{2m}\right)^2\prod_{m=1}^{n}\left(1+\frac{1}{(m-1/2)^2}\right)\right]$$
$$= \frac{1}{\pi}\,\cos(i\pi)$$
$$= \frac{\cosh\pi}{\pi},$$

where the first part of the limit is evaluated via the Wallis product as exhibited above and for the second part we use the product formula for the cosine,

$$\cos\pi z = \lim_{n\to\infty}\prod_{m=1}^{n}\left(1-\frac{z^2}{(m-1/2)^2}\right)$$

$$= \lim_{n\to\infty}\prod_{m=1}^{n}\left(\left(1-\frac{z}{m-1/2}\right)\left(1+\frac{z}{m-1/2}\right)\right)$$

for $z = i$. While this may seem to draw on more facts than Solution 3, we note that the Wallis product is only a special case of the product formula for the sine (coming from $\sin(\pi/2) = 1$), so that once you decide to start using product formulas this does not necessarily qualify as a "new fact," and the route leading directly to the product formula for the cosine has the advantage that it maintains more symmetry throughout the course of the calculation.

In fact, an equivalent of the Wallis product may be easily deduced by going further with Solution 2, as the following generalization shows. Indeed, this equivalent version is in the precise form needed to complete our transition to the product formula for the cosine given above.

Generalization: From Solution 2 one can notice that the only relevant fact about the roots of the quadratic expression is that they sum to 1 (note that in our notation above the quadratic is $m^2-m+5/4 = (m+z)(m+\bar{z})$ and thus its roots are $-z$ and $-\bar{z}$, with sum 1). Exploiting this observation opens the possibility of finding other interesting identities by choosing the coefficients appropriately. It can be seen easily that every quadratic whose roots sum to 1 has the form

$$p(x) = a(x^2 - x + r(1-r)), \text{ with } a \neq 0,$$

the two roots being r and $1-r$. Therefore, taking $a = 1$,

$$S(r) := \lim_{n\to\infty} n \prod_{m=1}^{n} \left(1 - \frac{1}{m} + \frac{r(1-r)}{m^2}\right) = \frac{1}{\Gamma(r)\Gamma(1-r)} = \frac{\sin(\pi r)}{\pi}$$

(note that to calculate $S(r)$ one can simply follow our work in Solution 2 above, with $-r$ and $-(1-r)$ replacing z and $\bar{z}$, and thus at the end with $1-r$ and $1-(1-r) = r$ replacing $z+1$ and $\bar{z}+1$). The proposed problem is the special case $r = \frac{1}{2} + i$ ($= z + 1 = -\bar{z}$, where $z = -\frac{1}{2} + i$). With $r = 1/2$, one has $S(1/2) = \frac{1}{\pi}$, i.e.,

$$\pi = \lim_{n\to\infty} \frac{1}{n} \prod_{m=1}^{n} \left(\frac{4m^2}{4m^2 - 4m + 1}\right).$$

This product representation may be written as

$$\pi = \lim_{n\to\infty} \frac{1}{n} \prod_{m=1}^{n} \left(\frac{(2m)(2m)}{(2m-1)(2m-1)}\right),$$

and the relation with the Wallis product is seen by direct comparison.

Other choices of r (such as $-1/2, 3/2, -3/2, \ldots$) also lead to closely related limit formulas for π, though with less symmetry. The reader is encouraged to experiment with these and other values of r (such as $1/3, 1/4, 1/6, \ldots$) as well.