An inequality.

11458. *Proposed by Cezar Lupu (student), University of Bucharest, Bucharest, Romania, and Vicentiu Radulescu, Institute of Mathematics "Simon Stoilow" of the Romanian Academy, Bucharest, Romania.* Let $a_1, \ldots, a_n$ be nonnegative and let r be a positive integer. Show that

$$\left(\sum_{1 \le i,j \le n} \frac{i^r j^r a_i a_j}{i+j-1} \right)^2 \le \sum_{m=1}^{n} m^{r-1} a_m \sum_{1 \le i,j,k \le n} \frac{i^r j^r k^r a_i a_j a_k}{i+j+k-2}.$$

Solution by Francisco Vial (student), MT group, Pontificia Universidad Católica de Chile, Santiago, Chile. We will use a generating function for the sequence $a_1, \ldots, a_n$ and Cauchy-Schwarz inequality. Set

$$f(x) := \sum_{i=1}^{n} i^r a_i x^{i-1}.$$

Then,

$$\begin{aligned}
\int_0^1 f(x)dx &= \sum_{i=1}^{n} \frac{i^r a_i}{i} = \sum_{m=1}^{n} m^{r-1} a_m, \\
\int_0^1 f(x)^2 dx &= \int_0^1 \left(\sum_{1 \le i,j \le n} i^r j^r a_i a_j x^{i+j-2} \right) dx = \sum_{1 \le i,j \le n} \frac{i^r j^r a_i a_j}{i+j-1}, \\
\int_0^1 f(x)^3 dx &= \int_0^1 \left(\sum_{1 \le i,j,k \le n} i^r j^r k^r a_i a_j a_k x^{i+j+k-3} \right) dx = \sum_{1 \le i,j,k \le n} \frac{i^r j^r k^r a_i a_j a_k}{i+j+k-2}.
\end{aligned}$$

Therefore, the inequality stated is equivalent to

$$\left(\int_0^1 f(x)^2 dx \right)^2 \le \left(\int_0^1 f(x) dx \right) \left(\int_0^1 f(x)^3 dx \right),$$

which follows from Cauchy-Schwarz inequality with the functions $f(x)^{1/2}, f(x)^{3/2}$ and the standard inner product (note that, as $a_i \ge 0$, $f(x) \ge 0$ in $[0,1]$, so these functions are real and well defined). ■

Remarks: "r" can be any real number, not necessarily an integer. In fact, the same inequality holds if we use any positive function on $[1,n]$ instead of i^r. For example, using the exponential function,

$$\left(\sum_{1 \le i,j \le n} \frac{e^{i+j} a_i a_j}{i+j-1} \right)^2 \le \sum_{m=1}^{n} \frac{e^m}{m} a_m \sum_{1 \le i,j,k \le n} \frac{e^{i+j+k} a_i a_j a_k}{i+j+k-2}.$$