11459. *Proposed by Pál Péter Dályay, Deák Ferenc High School, Szeged, Hungary.* Find all pairs (s,z) of complex numbers such that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{k!(n-k)!}\left(\prod_{j=1}^{k}(sj-z)\right)\left(\prod_{j=0}^{n-k-1}(sj+z)\right)$$

converges.

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The answer is that the pair (s,z) is constrained only by the condition $|s|<1$, with z arbitrary. In fact, we show that the given series reduces to $\sum_{n=0}^{\infty} s^n$. Since this series converges for $|s|<1$ and diverges otherwise, this proves our conclusion. In this case the sum of the series is given succinctly as $(1-s)^{-1}$. We give two proofs of the result.

First Solution: Let $S(s,z)$ be the given series. We repackage the product terms so that they can be identified with binomial coefficients. This will necessitate factoring an s out of each term in each of the products, and forces us to consider the case $s=0$ separately (and much the same is needed in our second solution, as will be seen below).

Recall that the binomial coefficient $\binom{\alpha}{k}$ (where α is an arbitrary complex number and k is a nonnegative integer) represents the expression $\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)/k!$ (with $\binom{\alpha}{0}$ always taken to be 1, i.e., "empty products" are taken as 1). Its first key property, and the reason for its name, is the general binomial expansion

$$\sum_{n=0}^{\infty}\binom{\alpha}{n}x^n=(1+x)^{\alpha}.$$

Then, assuming $s\neq 0$, the product terms can be rewritten using

$$\frac{1}{k!}\prod_{j=1}^{k}(sj-z)=(-s)^k\frac{1}{k!}\Big(\frac{z}{s}-1\Big)\Big(\frac{z}{s}-2\Big)\cdots\Big(\frac{z}{s}-k\Big)=(-s)^k\binom{\frac{z}{s}-1}{k}$$

and

$$\begin{aligned}\frac{1}{(n-k)!}\prod_{j=0}^{n-k-1}(sj+z) &= (-s)^{n-k}\frac{1}{(n-k)!}\Big(-\frac{z}{s}\Big)\Big(-\frac{z}{s}-1\Big)\cdots\Big(-\frac{z}{s}-(n-k-1)\Big)\\ &= (-s)^{n-k}\binom{-\frac{z}{s}}{n-k}\end{aligned}$$

and thus our inner sum becomes

$$(-s)^n\sum_{k=0}^{n}\binom{\frac{z}{s}-1}{k}\binom{-\frac{z}{s}}{n-k}.$$

But the sum appearing here is exactly a special case of the well-known Chu–Vandermonde identity for binomial coefficients (see Remark) which says

$$\sum_{k=0}^{n}\binom{\alpha}{k}\binom{\beta}{n-k}=\binom{\alpha+\beta}{n},$$

for all complex numbers α and β. Thus our inner sum evaluates to $(-s)^n\binom{-1}{n}$ and therefore

$$S(s,z)=\sum_{n=0}^{\infty}\binom{-1}{n}(-s)^n=(1-s)^{-1},$$

where we have used the binomial expansion given above (alternatively, one could observe that $\binom{-1}{n}=(-1)^n n!/n!=(-1)^n$ and use the fact that this reduces the sum in question to a geometric series). For the case of what happens when $s = 0$, one could appeal to a continuity argument to convince oneself that the answer is 1 ($= (1-s)^{-1}|_{s=0}$), but perhaps it is more instructive to make a direct argument, since in this case the argument simplifies considerably. When $s = 0$ the two products reduce simply to $(-z)^k$ and z^{n-k} and the inner sum is then a simple binomial sum giving $(z-z)^n$, which of course is 1 for $n = 0$ and 0 for all larger values of n. Thus, doing the outer sum gives the answer 1.

Second Solution: Let $S(s,z)$ be the given series. For $s \neq 0$, we have

$$\frac{1}{k!(n-k)!}\left(\prod_{j=1}^{k}(sj-z)\right)\left(\prod_{j=0}^{n-k-1}(sj+z)\right) = \frac{(-s)^n}{n!}\binom{n}{k}\left(\prod_{j=1}^{k}\left(\frac{z}{s}-j\right)\right)\left(\prod_{j=0}^{n-k-1}\left(-\frac{z}{s}-j\right)\right).$$

We now write the products on the right as the following derivatives:

$$\prod_{j=1}^{k}\left(\frac{z}{s}-j\right) = \frac{d^k}{dx^k}\left(x^{z/s-1}\right)\Big|_{x=1}, \qquad \prod_{j=0}^{n-k-1}\left(-\frac{z}{s}-j\right) = \frac{d^{n-k}}{dx^{n-k}}\left(x^{-z/s}\right)\Big|_{x=1}.$$

Inserting these expressions into $S(s,z)$ we find

$$\begin{aligned} S(s,z) &= \sum_{n=0}^{\infty}\frac{(-s)^n}{n!}\left(\sum_{k=0}^{n}\binom{n}{k}\frac{d^k}{dx^k}\left(x^{z/s-1}\right)\frac{d^{n-k}}{dx^{n-k}}\left(x^{-z/s}\right)\right)\Bigg|_{x=1} \\ &= \sum_{n=0}^{\infty}\frac{(-s)^n}{n!}\,\frac{d^n}{dx^n}\left(x^{-1}\right)\Big|_{x=1} \end{aligned}$$

where we have used Leibniz's rule for differentiation of a product to collapse the inner sum ro a single term. We then have

$$S(s,z) = \sum_{n=0}^{\infty}\frac{(-s)^n}{n!}(-1)^n n! = \sum_{n=0}^{\infty} s^n = (1-s)^{-1}$$

for $|s| < 1$.

The $s = 0$ case has already been treated in the first solution.

Remark: One proof of the Chu-Vandermonde identity for binomial coefficients is to multiply the binomial expansions for $(1+s)^\alpha$ and $(1+s)^\beta$, observing on the one hand that the result is $(1+s)^{\alpha+\beta}$ (and in turn representing this as a power series in s via its binomial expansion), and, on the other, multiplying out the binomial expansions for the α and β series and using the Cauchy product formula to collect the coefficients in the expansion of the product. Then identifying the coefficients in the two expansions gives the Vandermonde identity.
Indeed, the Cauchy product formula for the expansion of the product of two series (not necessarily power series) gives yet another way of evaluating the given series. Without getting into questions of convergence (which are not difficult to handle) one has (where we adopt the binomial coefficient notation introduced earlier), for $s \neq 0$,

$$S(s,z) = \left(\sum_{n=0}^{\infty}\frac{1}{n!}\prod_{j=1}^{n}(sj-z)\right)\left(\sum_{n=0}^{\infty}\frac{1}{n!}\prod_{j=0}^{n-1}(sj+z)\right) = (1-s)^{z/s-1}(1-s)^{-z/s} = (1-s)^{-1},$$

where in the second to last line we have reduced the two infinite sums to closed form binomials using the binomial expansion. The case where $s = 0$ can be dealt with in like fashion, yielding $e^{-z}e^{z}$, or 1.