1210. *Proposed by Robert Gebhardt, Hopatcong, NJ.*

Find an equation for the plane tangent to the ellipsoid $a^2x^2 + b^2y^2 + c^2z^2 = d^2$ in the first octant such that the volume in the first octant bounded by this plane and the coordinate planes is minimum.

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The equation for the tangent plane that minimizes the given volume and the value of the minimal volume are given respectively by

$$\Pi: \ ax + by + cz = d\sqrt{3}, \qquad V = \frac{d^3}{abc}\frac{\sqrt{3}}{2}.$$

We are of course considering $a, b, c, d \neq 0$ (if $a = 0$ or $b = 0$ or $c = 0$, there is no minimizing plane, and if $d = 0$, the answer is trivial because the ellipsoid is the origin). Moreover, without loss of generality we have assumed $a, b, c, d > 0$.

It is well-known and straightforward to prove that the equation for the plane tangent to the ellipsoid at (x_0, y_0, z_0), a point lying on the ellipsoid, is

$$a^2x_0x + b^2y_0y + c^2z_0z = d^2.$$

To prove the latter, it suffices to compute the gradient of $f(x, y, z) = a^2x^2 + b^2y^2 + c^2z^2$, which is normal to the surface $f = d^2$ at (x, y, z), and then write the equation of the plane as $(\vec{r} - \vec{r}_0) \cdot \hat{n} = 0$, where $\vec{r} := (x, y, z)$. Computing the intersection between the plane and each coordinate axis, one has that the volume is given by

$$V(x_0, y_0, z_0) = \frac{1}{6}\frac{d^6}{(a^2x_0)(b^2y_0)(c^2z_0)}.$$

(I.e., width$\times$length$\times$height$/6$). Thus, using the arithmetic-geometric (henceforth AM-GM) inequality,

$$\begin{aligned} V &= \frac{1}{6}\frac{d^6}{(a^2x_0)(b^2y_0)(c^2z_0)} = \frac{d^6}{6abc}\left(\frac{1}{\sqrt[3]{(ax_0)^2(by_0)^2(cz_0)^2}}\right)^{3/2} \geq \frac{d^6}{6abc}\left(\frac{3}{a^2x_0^2 + b^2y_0^2 + c^2z_0^2}\right)^{3/2} \\ &= \frac{d^6}{6abc}\frac{3^{3/2}}{d^3} = \frac{d^3}{abc}\frac{\sqrt{3}}{2}, \end{aligned}$$

where we have used the fact that $a^2x_0^2 + b^2y_0^2 + c^2z_0^2 = d^2$. Equality in AM-GM is reached if and only if

$$(ax_0)^2 = (by_0)^2 = (cz_0)^2.$$

Replacing into $a^2x_0^2 + b^2y_0^2 + c^2z_0^2 = d^2$ and recalling that (x_0, y_0, z_0) must lie in the first octant, yields

$$x_0 = \frac{1}{\sqrt{3}}\frac{d}{a}, \quad y_0 = \frac{1}{\sqrt{3}}\frac{d}{b}, \quad z_0 = \frac{1}{\sqrt{3}}\frac{d}{c}.$$

Plugging this in the plane equation and simplifying gives

$$ax + by + cz = d\sqrt{3}. \quad \blacksquare$$

We claim that the following generalization holds: Consider the n–dimensional ellipsoid given by the equation

$$\sum_{k=1}^{n} a_k^2 x_k^2 = a^2,$$

with $a_k > 0$ for $k = 1, 2, \ldots, n$. The tangent hyperplane in the "first octant" that minimizes the $\mathbb{R}^{n-1}$ measure of the set bounded by the hyperplane and the coordinate hyperplanes is given by the equation

$$\sum_{k=1}^{n} a_k x_k = a\sqrt{n},$$

and the minimum volume is given by

$$V = \frac{a^n}{P}\frac{n^{n/2}}{n!},$$

where $P = a_1 a_2 \cdots a_n$.

Proof: in a similar fashion, one wants to minimize the volume of the simplex, given by

$$V(\vec{x}) = \frac{1}{n!}\prod_{k=1}^{n}\frac{a^2}{a_k^2 x_k},$$

with the constraints $x_k > 0$ for $k = 1, 2, \ldots, n$ and $\sum_{k=1}^{n} a_k^2 x_k^2 = a^2$. Then

$$\begin{aligned} V(\vec{x}) &= \frac{1}{n!}\frac{a^{2n}}{P}\prod_{k=1}^{n}\frac{1}{a_k x_k} = \frac{1}{n!}\frac{a^{2n}}{P}\left(\prod_{k=1}^{n}\frac{1}{\sqrt[n]{a_k^2 x_k^2}}\right)^{n/2} \\ &\geq \frac{1}{n!}\frac{a^{2n}}{P}\left(\frac{n}{\sum_{k=1}^{n} a_k^2 x_k^2}\right)^{n/2} = \frac{1}{n!}\frac{a^{2n}}{P}\frac{n^{n/2}}{a^n} = \frac{a^n}{P}\frac{n^{n/2}}{n!}, \end{aligned}$$

where we have used the $AM - GM$ inequality. The minimum is reached (in the first octant) if and only if $a_1^2 x_1^2 = a_2^2 x_2^2 = \cdots = a_n^2 x_n^2$. Substituting this into the ellipsoid equation then yields the coordenates of the "minimizer point":

$$x_k = \frac{1}{\sqrt{n}}\frac{a}{a_k},\ k = 1, 2, \ldots, n.$$

Replacing into the plane equation and simplifying proves the claim.