Floor function Sum.

1786. *Proposed by Bob Tomper, University of North Dakota, Grand Forks, ND.*

Let m and n be relatively prime positive integers. Prove that

$$\sum_{k=1}^{n} k^2 \left\lfloor \frac{km}{n} \right\rfloor = n \sum_{k=1}^{n} k \left\lfloor \frac{km}{n} \right\rfloor - \frac{n(n^2-1)(m-1)}{12}.$$

Solution by Francisco Vial (student), Pontificia Universidad Católica de Chile, Santiago de Chile.

Let $f_n(k) = (nk - k^2) \left\lfloor \frac{km}{n} \right\rfloor$ and

$$S := \sum_{k=1}^{n} f_n(k) = \sum_{k=1}^{n} (nk - k^2) \left\lfloor \frac{km}{n} \right\rfloor = \sum_{k=0}^{n} (nk - k^2) \left\lfloor \frac{km}{n} \right\rfloor.$$

Then,

$$\begin{aligned}
2S &= \sum_{k=0}^{n} f_n(k) + \sum_{k=0}^{n} f_n(n-k) \\
&= \sum_{k=0}^{n} (nk - k^2) \left\lfloor \frac{km}{n} \right\rfloor + \sum_{k=0}^{n} (n(n-k) - (n-k)^2) \left\lfloor \frac{(n-k)m}{n} \right\rfloor \\
&= \sum_{k=0}^{n} (nk - k^2) \left\lfloor \frac{km}{n} \right\rfloor + \sum_{k=0}^{n} (nk - k^2) \left\lfloor \frac{(n-k)m}{n} \right\rfloor \\
&= \sum_{k=0}^{n} (nk - k^2) \left(\left\lfloor \frac{km}{n} \right\rfloor + \left\lfloor \frac{(n-k)m}{n} \right\rfloor \right).
\end{aligned}$$

Notice that

$$\left\lfloor \frac{(n-k)m}{n} \right\rfloor = \left\lfloor m - \frac{km}{n} \right\rfloor = m + \left\lfloor -\frac{km}{n} \right\rfloor,$$

since m is integer. Hence,

$$S = \frac{1}{2} \sum_{k=0}^{n} (nk - k^2) \left(m + \left\lfloor \frac{km}{n} \right\rfloor + \left\lfloor -\frac{km}{n} \right\rfloor \right).$$

Given $(m, n) = 1$, the number $x_k := \frac{m}{n} k$ is non-integer for all $0 < k < n$, in which case the following holds:

$$\lfloor x_k \rfloor + \lfloor -x_k \rfloor = -1.$$

If $k = n$, we have

$$\left\lfloor \frac{nm}{n} \right\rfloor + \left\lfloor \frac{-nm}{n} \right\rfloor = 0.$$

Then

$$\begin{aligned}
S &= \frac{1}{2} \sum_{k=1}^{n-1} (nk - k^2)(m-1) + \frac{1}{2}(0n - 0^2)(m) + \frac{1}{2}(nn - n^2)(m) \\
&= \frac{m-1}{2} \sum_{k=1}^{n-1} nk - k^2 \\
&= \frac{(m-1)}{2} \left(\frac{n^2(n-1)}{2} - \frac{n(2n-1)(n-1)}{6} \right), \\
S &= \frac{n(n^2-1)(m-1)}{12}.
\end{aligned}$$

Finally

$$\begin{aligned}
\sum_{k=1}^{n}(nk-k^2)\left\lfloor\frac{km}{n}\right\rfloor &= \frac{n(n^2-1)(m-1)}{12}, \\
\sum_{k=1}^{n}k^2\left\lfloor\frac{km}{n}\right\rfloor &= n\sum_{k=1}^{n}k\left\lfloor\frac{km}{n}\right\rfloor - \frac{n(n^2-1)(m-1)}{12}.
\end{aligned}$$

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