An integer matrix with permutations.

1827. *Proposed by Christopher Hilliar, Texas A&M University, College Station, TX.*

Let A be a $n \times n$ matrix with integer entries and such that each column of A is a permutation of the first column. Prove that if the entries in the first column does not sum to 0, then this sum divides $\det(A)$.

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First Solution

Suppose that $\det(A) \neq 0$ (if $\det(A) = 0$ there is nothing to prove). Let s be the sum of the elements of the first column. As the other columns are permutations of the first one, they all sum to s. Then, it is easy to see that if one sums all the rows to the first one, the result will be $s, s, s, \ldots, s$ in the first row, so factoring s out of the determinant proves the claim, since the remaining determinant has integer entries with 1's in the first row (i.e., its value is an integer number).

Second Solution

Suppose that $\det(A) \neq 0$ (if $\det(A) = 0$ there is nothing to prove). Let s be the sum of the elements of the first column. As the other columns are permutations of the first one, they all sum to s. Consider now A^T, the transposed matrix of A. Then the sum of each row of A^T is s. If one performs the product $A^T(1,1,1,\ldots,1)^T$, then the result is evidently $(s,s,s,\ldots,s) = s(1,1,1,\ldots,1)$, i.e., s is an eigenvalue of A^T. Let μ_i be the eigenvalues of A^T, with $\mu_n = s$, then

$$\mu_1 \cdot \mu_2 \cdot \cdots \cdot \mu_{n-1} s = \det(A^T) = \det(A).$$

We will now prove that the product $\mu_1 \cdot \mu_2 \cdot \cdots \cdot \mu_{n-1}$ is an integer number. Consider the characteristic polynomial of A^T, given by

$$P(x) := |A^T - xI| = (\mu_1 - x)(\mu_2 - x)\ldots(s - x).$$

From the first representation one reads that this polynomial has integer coefficients. The coefficient of $(-x)^{n-1}$ is

$$C_{n-1} = \mu_1 + \mu_2 + \cdots + \mu_{n-1} + s$$

i.e., $\mu_1 + \mu_2 + \cdots + \mu_{n-1}$ is an integer.

The coefficient of $(-x)^{n-2}$ is

$$\begin{aligned} C_{n-2} &= \sum_{1 \le i < j \le n} \mu_i \mu_j = s\mu_1 + s\mu_2 + \cdots + s\mu_{n-1} + \sum_{1 \le i < j \le n-1} \mu_i \mu_j \\ &= s(\mu_1 + \mu_2 + \cdots + \mu_{n-1}) + \sum_{1 \le i < j \le n-1} \mu_i \mu_j, \end{aligned}$$

i.e., $\sum_{1 \le i < j \le n-1} \mu_i \mu_j$ is an integer.

Moving one step further, one could show, using the last result, that $\sum \mu_i \mu_j \mu_k$, with $1 \le i < j < k \le n-1$, is an integer.

By induction it is straighforward to prove that for every natural k, $k \le n-1$,

$$\sum \mu_{i_1} \mu_{i_2} \cdots \mu_{i_k}$$

where the sum is cyclic and the i_m's are all distinct, is an integer number. Taking $k = n-1$,

$$\sum \mu_{i_1} \mu_{i_2} \cdots \mu_{i_{n-1}} = \mu_1 \mu_2 \cdots \mu_{n-1}$$

is an integer number M, and the conclusion follows:

$$Ms = \det(A). \qquad \blacksquare$$

Remark: We have only used the facts that all the columns sum to s and that the matrix has integer entries. The permutation hypothesis is not necessary.