Pell-Lucas Numbers

907. *Proposed by Brian Bradie, Christopher Newport University, Newport News, VA.*

For each nonnegative integer n, let

$$a_n = \left(\sum_{j=0}^{n} Q_j\right)^2 - \sum_{j=0}^{n} Q_{2j+1},$$

where Q_n is the nth Pell-Lucas number, that is, $Q_0 = 2$, $Q_1 = 2$, and $Q_n = 2Q_{n-1} + Q_{n-2}$ for $n \geq 2$. Evaluate

$$\sum_{n=0}^{\infty} \frac{a_n}{n!}.$$

Solution by Francisco Vial (student), Pontificia Universidad Católica de Chile, Santiago, Chile. The answer is $2\cosh 1$, as we will explain here. The characteristic polynomial of the Pell-Lucas sequence is

$$r^n = 2r^{n-1} + r^{n-2},$$

where the roots are $r = 1 \pm \sqrt{2}$. Given $Q_0 = Q_1 = 2$, the general term is

$$Q_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n.$$

Use the geometric series to compute the sum

$$\begin{aligned}\sum_{j=0}^{n} Q_j &= \sum_{j=0}^{n}(1+\sqrt{2})^j + (1-\sqrt{2})^j = \sum_{j=0}^{n}(1+\sqrt{2})^j + \sum_{j=0}^{n}(1-\sqrt{2})^j \\ &= \frac{(1+\sqrt{2})^{n+1} - 1}{\sqrt{2}} + \frac{1-(1-\sqrt{2})^{n+1}}{\sqrt{2}} \\ &= \frac{1}{\sqrt{2}}\left((1+\sqrt{2})^{n+1} - (1-\sqrt{2})^{n+1}\right).\end{aligned}$$

Upon squaring this sum,

$$\begin{aligned}\left(\sum_{j=0}^{n} Q_j\right)^2 &= \frac{1}{2}\left((1+\sqrt{2})^{2n+2} - 2(1+\sqrt{2})^{n+1}(1-\sqrt{2})^{n+1} + (1-\sqrt{2})^{2n+2}\right) \\ &= \frac{Q_{2n+2}}{2} - (-1)^{n+1} = \frac{Q_{2n+2}}{2} + (-1)^n.\end{aligned}$$

Besides, from the recursive relation,

$$\begin{aligned}\frac{Q_{2n+2}}{2} &= Q_{2n+1} + \frac{Q_{2n}}{2} = Q_{2n+1} + Q_{2n-1} + \frac{Q_{2n-2}}{2} = Q_{2n+1} + Q_{2n-1} + Q_{2n-3} + \frac{Q_{2n-4}}{2} = \dots \\ &= Q_{2n+1} + Q_{2n-1} + \cdots + Q_1 + \frac{Q_0}{2} = \sum_{j=0}^{n} Q_{2j+1} + \frac{Q_0}{2}.\end{aligned}$$

Hence,

$$\sum_{j=0}^{n} Q_{2j+1} = \frac{Q_{2n+2}}{2} - \frac{Q_0}{2} = \frac{Q_{2n+2}}{2} - 1$$

This calculations give a general term for a_n

$$\begin{aligned}
a_n &= \left(\sum_{j=0}^{n} Q_j\right)^2 - \sum_{j=0}^{n} Q_{2j+1} \\
&= (\frac{Q_{2n+2}}{2} + (-1)^n) - (\frac{Q_{2n+2}}{2} - 1) \\
&= (-1)^n + 1.
\end{aligned}$$

Finally,

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{a_n}{n!} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} + \sum_{n=0}^{\infty} \frac{1}{n!} \\
&= e^{-1} + e = 2\cosh 1.
\end{aligned}$$