1217. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Calculate

$$\int_0^1 \left\{ (-1)^{\lfloor \frac{1}{x} \rfloor} \cdot \frac{1}{x} \right\} dx,$$

where $\lfloor a \rfloor$ denotes the floor of a and $\{a\} = a - \lfloor a \rfloor$ denotes the fractional part of a.

Solution by Francisco Vial (student), Pontificia Universidad Católica de Chile, Santiago de Chile. The answer is ???, where γ is the Euler-Mascheroni constant.

Let I be the value of the integral given. For any positive integer k, let $I_k = (\frac{1}{k+1}, \frac{1}{k})$. We have that if $x \in (\frac{1}{k+1}, \frac{1}{k})$, then $\frac{1}{x} \in (k, k+1)$, therefore $\lfloor \frac{1}{x} \rfloor = k$. Write

$$I = \sum_{k \geq 1} \int_{I_k} \left\{ (-1)^{\lfloor \frac{1}{x} \rfloor} \cdot \frac{1}{x} \right\} dx = \sum_{k \geq 1} \int_{I_k} \left\{ (-1)^k \cdot \frac{1}{x} \right\} dx.$$

We now consider a partial sum I_N and split it into two parts:

$$\begin{aligned} I_N &= \sum_{k=1}^N \int_{I_k} \left\{ (-1)^k \cdot \frac{1}{x} \right\} = \sum_{k=1, k \text{ odd}}^N \int_{I_k} \left\{ (-1)^k \cdot \frac{1}{x} \right\} + \sum_{k=1, k \text{ even}}^N \int_{I_k} \left\{ (-1)^k \cdot \frac{1}{x} \right\} \\ &= \sum_{k=1, k \text{ odd}}^N \int_{I_k} \left\{ -\frac{1}{x} \right\} + \sum_{k=1, k \text{ even}}^N \int_{I_k} \left\{ \frac{1}{x} \right\} \end{aligned}$$

We analyse the "even" partial sum, of even order (in order to get asymptotics and simplify the computations). By direct integration,

$$\int_{1/(k+1)}^{1/k} \left\{ \frac{1}{x} \right\} dx = \int_{1/(k+1)}^{1/k} \left(\frac{1}{x} - k \right) dx = \ln \left(\frac{k+1}{k} \right) - \frac{1}{k+1},$$

hence

$$S_{even}^{(2N)} = \sum_{k=1, k \text{ even}}^{2N} \left(\ln \left(\frac{k+1}{k} \right) - \frac{1}{k+1} \right) = \sum_{k=1}^N \ln \left(\frac{2k+1}{2k} \right) - \sum_{k=1}^N \frac{1}{2k+1}.$$

For the first sum we have

$$\sum_{k=1}^N \ln \left(\frac{2k+1}{2k} \right) = \ln \prod_{k=1}^N \frac{k+1/2}{k} = \ln \frac{\Gamma(N+3/2)\Gamma(1)}{\Gamma(3/2)\Gamma(N+1)} \sim \ln \frac{\Gamma(1)}{\Gamma(3/2)} + \frac{1}{2} \ln N + O\left(\frac{1}{N} \right).$$

where we have used the factorial property of the Gamma function, and the fact that $\Gamma(z+a)/\Gamma(z+b) \sim z^{a-b} + o(z^{a-b-1})$ as $|z| \to \infty$ (see, for instance, Abramowitz and Stegun: "Handbook of Mathematical Functions", pp.257, equation 6.1.47). With this,

$$S_{even}^{(2N)} \sim \frac{1}{2} \ln N - \ln \Gamma(3/2) - \sum_{k=1}^N \frac{1}{2k+1} + O\left(\frac{1}{N} \right).$$

We repeat the argument for the "odd" sum of odd order. As $\{x\} + \{-x\} = 1$ if $x \notin \mathbb{Z}$, by direct integration one has

$$\int_{1/(k+1)}^{1/k} \left\{ -\frac{1}{x} \right\} dx = \int_{1/(k+1)}^{1/k} \left(1 - \left\{ \frac{1}{x} \right\} \right) dx = \frac{1}{k} - \ln \frac{k+1}{k}.$$

$$S_{odd}^{(2N-1)} = \sum_{k=1, k \text{ odd}}^{2N-1} \left(\frac{1}{k} - \ln \frac{k+1}{k} \right) = \sum_{k=1}^{N} \frac{1}{2k-1} - \sum_{k=1}^{N} \ln \left(\frac{2k}{2k-1} \right).$$

By the arguments presented earlier,

$$\sum_{k=1}^{N} \ln \left(\frac{k}{k-1/2} \right) = \ln \frac{\Gamma(N+1)\Gamma(1/2)}{\Gamma(1)\Gamma(N+1/2)} \sim \ln \frac{\Gamma(1/2)}{\Gamma(1)} + \frac{1}{2} \ln(N) + O\left(\frac{1}{N} \right),$$

and therefore

$$S_{odd}^{(2N-1)} \sim \sum_{k=1}^{N} \frac{1}{2k-1} - \ln \Gamma(1/2) - \frac{1}{2} \ln(N) + O\left(\frac{1}{N} \right)$$

Finally,

$$\begin{aligned} I_{2N} &= S_{odd}^{(2N-1)} + S_{even}^{2N} \sim \sum_{k=1}^{N} \left(\frac{1}{2k-1} - \frac{1}{2k+1} \right) - \ln(\Gamma(1/2)\Gamma(3/2)) + O\left(\frac{1}{N} \right) \\ &= 1 - \frac{1}{2N+1} - \ln(\Gamma(1/2)\Gamma(3/2)) + O\left(\frac{1}{N} \right) \to 1 - \ln(\Gamma(1/2)\Gamma(3/2)). \end{aligned}$$

as $N \to \infty$. For the evaluation of the logarithm, it suffices to know the values of $\Gamma(1/2) = \sqrt{\pi}$ and the recursive relation $x\Gamma(x) = \Gamma(x+1)$, giving $\Gamma(1/2)\Gamma(3/2) = \frac{1}{2}\Gamma(1/2)^2 = \pi/2$.

By the evident convergence of the sequence I_N (which can be justified by remarking that it converges to the convergent integral posed, or by means of series comparison criteria), we have that

$$I = \lim_{N \to \infty} I_N = \lim_{N \to \infty} I_{2N} = 1 - \ln \frac{\pi}{2}.$$