1828. *Proposed by Stephen J. Herschkorn, Department of Statistics, Rutgers University, New Brunswick, NJ.*

Let α_0 be the smallest value of α for which exists a positive constant C such that

$$\prod_{k=1}^{n} \frac{2k}{2k-1} \leq Cn^{\alpha}$$

for all positive integer n.

a. Find the value of α_0.

b. Prove that the sequence

$$\left\{ \frac{1}{n^{\alpha_0}} \prod_{k=1}^{n} \frac{2k}{2k-1} \right\}_{n=1}^{\infty}$$

is decreasing and find its limit.

Solution by Mark Ashbaugh, University of Missouri, Columbia, and Francisco Vial (student), Pontificia Universidad Católica de Chile, Santiago de Chile.

a) The inequality given is satisfied if and only if the sequence

$$u_n := \frac{1}{n^{\alpha}} \prod_{k=1}^{n} \frac{2k}{2k-1}$$

is bounded. A suficient condition for this is that it converges. We write

$$u_n := \frac{1}{n^{\alpha}} \prod_{k=1}^{n} \frac{(2k)(2k)}{(2k-1)(2k)} = \frac{1}{n^{\alpha}} \frac{2^{2n} n!^2}{(2n)!},$$

and Stirling's aproximation for the factorial states $n! = n^n \sqrt{2\pi n} e^{-n}(1 + O(1/n))$ as n approaches ∞. Using this in the latter expression one finds

$$u_n \sim \frac{1}{n^{\alpha}} \frac{2^{2n} \left(n^n \sqrt{2\pi n} e^{-n})\right)^2}{(2n)^{2n} \sqrt{4\pi n} e^{-2n}} = \sqrt{\pi} n^{1/2-\alpha}.$$

One concludes that if $\alpha < 1/2$, u_n is not bounded. On the other hand, if $\alpha \geq 1/2$, u_n is convergent and thus bounded. Therefore, $\alpha_0 = 1/2$.

b) (First Solution)

We first write the product given in terms of factorials, as before:

$$\prod_{k=1}^{n} \frac{2k}{2k-1} = \frac{2^{2n}(n!)^2}{(2n)!},$$

Taking $\alpha_0 = 1/2$, let us denote the sequence given as

$$x_n := \frac{1}{\sqrt{n}} \frac{2^{2n}(n!)^2}{(2n)!}.$$

A straightforward calculation yields

$$\begin{aligned} \frac{x_{n+1}}{x_n} &= \frac{\sqrt{n(n+1)}}{n+1/2} \\ &< \frac{\sqrt{n(n+1)+\frac{1}{4}}}{n+1/2} = 1, \end{aligned}$$

which allows to conclude that the sequence is strictly decreasing.

Using Stirling's formula, as above we have

$$x_n \sim \frac{1}{\sqrt{n}}\sqrt{\pi n} = \sqrt{\pi}.$$

as n approaches $+\infty$, therefore, $\lim x_n = \sqrt{\pi}$.

b) *(Second Solution)*

Let (x_n) be the sequence given (the proof that (x_n) is decreasing is as in the first solution). Then

$$x_n = \frac{1}{\sqrt{n}}\frac{(2n)!!}{(2n-1)!!} = \frac{\sqrt{\pi}}{\sqrt{n}}\frac{2^n\Gamma(n+1)}{2^n\Gamma(n+1/2)} = \frac{\sqrt{\pi}}{\sqrt{n}}\frac{\Gamma(n+1)}{\Gamma(n+1/2)},$$

We have also that the Gamma function has the property that $\Gamma(x+a)/\Gamma(x+b)$ goes as x^{a-b} as x goes to $+\infty$ for a and b fixed (complex) numbers (see, for example, F. W. J. Olver, Asymptotics and Special Functions, Academic Press, 1974, pp. 118-119, or G. E. Andrews, R. Askey, and R. Roy, Special Functions, Cambridge University Press, Cambridge, 1999, pp. 615–616). Thus, $\Gamma(n+1)/\Gamma(n+1/2)$ goes like $\sqrt{n}$ as n goes to $+\infty$ and therefore

$$\lim_{n\to\infty} x_n = \sqrt{\pi}\lim_{n\to\infty}\frac{1}{\sqrt{n}}\frac{\Gamma(n+1)}{\Gamma(n+1/2)} = \sqrt{\pi}.$$

Remark: This exact problem can be seen in Harry Hochstadt's *The Functions of Mathematical Physics* (Dover, 1986; originally published by Wiley in 1971), see prob. 15 on page 166 (unfortunately, the problem as stated has some misprints, but it should read "Show that if . . . , then $\sqrt{n}\,u_n$ is an increasing sequence . . .").

In fact, this has interesting implications for problem 16 (page 167), and this concerns bounds for the Legendre polynomials, namely

$$\sqrt{\sin\theta}|P_n(\cos(\theta))| \le \sqrt{\frac{2}{\pi n}},\ \ 0 < \theta \le \pi,\ \ n = 1, 2, 3, ...$$

An interesting improvement to this bound is to find the largest real number β such that

$$\frac{1}{\sqrt{n+\beta}}\prod_{k=1}^{n}\frac{2k}{2k-1}$$

is decreasing.

The answer is $\beta = 1/4$, and this improves Hochstadt's result:

$$\sqrt{\sin\theta}|P_n(\cos(\theta))| \le \sqrt{\frac{2}{\pi(n+1/2)}},\ \ 0 \le \theta \le \pi,\ n = 0, 1, 2, ...$$

Remark 2: We present yet another way of proving the result. The reader is encouraged to prove the following:

Let $z \in \mathbb{C}$. Then

$$\frac{\sin(\pi z)}{\pi} = \lim_{n\to\infty}\frac{1}{n}\prod_{k=1}^{n}\left(1 - \frac{1}{k} + \frac{z(1-z)}{k^2}\right).$$

Taking $z = 1/2$ yields

$$\frac{1}{\pi} = \lim_{n\to\infty}\frac{1}{n}\prod_{k=1}^{n}\frac{4k^2}{4k^2-4k+1}.$$

Noticing $4m^2 - 4m + 1 = (2m-1)^2$, and extracting square roots allows to conclude.

For proving the latter representation, one can notice that the polynomial $p(x) = x^2 - x + r(1-r)$ has two simple roots r and $1-r$, whose sum is 1. Then, one can factorize the inner expression, write the product with Gamma functions, and finally use the reflection formula (or the assymptotics for $\Gamma(z+a)/\Gamma(z+b)$ presented earlier).

Taking other values for r, such as $r = -1/2, 3/2...$, gives other formulas for π.

Other way to solve this problem is to consider the Wallis product for π:

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{2n}{2n-1} \cdot \frac{2n}{2n+1}.$$

If one considers

$$u_n = \frac{1}{\sqrt{n}} \prod_{k=1}^{n} \frac{2k}{2k-1},$$

then the relation between u_n^2 and the above partial products for π, is seen by direct comparison.

Remark 3: We write

$$\prod_{k=1}^{n} \frac{2k-1}{2k} = \frac{2^{2n}(n!)^2}{(2n)!} = \frac{1}{4^n}\binom{2n}{n},$$

so we are basically considering here the asymptotics of the "central binomial coeficient". Consideration of this is maybe what led de Moivre to Stirling's formula (de Moivre is really the first to have proved Stirling's formula, and he did all the basic work, but he left the constant $\sqrt{2\pi}$ undetermined).

There are other applications as well of studying $\binom{2n}{n}$ in number theory. One is an approach to a (weak version of) the Chebyshev result for $\pi(n)$, the counting function for the primes. The other is an approach to Bertrand's theorem (for any integer $n \geq 2$ there must be a prime between n and $2n$). These discussions can be found in Underwood Dudley, Elementary Number Theory, 2nd ed., Dover, 2008 (see Sect. 21, pp. 163-171 and pp. 177-179).

Remark 4: We present another way to prove the result. This is similar to Remark 2, in the sense that a product representation for $\sin \pi x$ is required.

Consider the two sequences

$$p_n := \sqrt{n} \prod_{k=1}^{n} \frac{2k-1}{2k}, \qquad q_n := \frac{1}{\sqrt{n}} \prod_{k=1}^{n} \frac{2k+1}{2k}.$$

One can show that this are two convergent sequences. Moreover, let $p = \lim p_n, q = \lim q_n$. We may write

$$p_n = \sqrt{n} \prod_{k=1}^{n} \left(1 - \frac{x}{k}\right), \qquad q_n = \frac{1}{\sqrt{n}} \prod_{k=1}^{n} \left(1 + \frac{x}{k}\right),$$

with $x = 1/2$. We then have

$$p_n q_n = \prod_{k=1}^{n} \left(1 - \frac{x^2}{k^2}\right) \to \frac{\sin \pi x}{\pi x}$$

as $n \to \infty$. Also,

$$\frac{p_n}{q_n} = n \prod_{k=1}^{n} \frac{2k-1}{2k+1} = \frac{n}{2n+1} \to \frac{1}{2},$$

as $n \to \infty$. We then have

$$pq = \frac{2}{\pi}, \qquad p/q = \frac{1}{2}.$$

Multiplying both equations yields $p = 1/\sqrt{\pi}$, the result desired. Also, $q = 2/\sqrt{\pi}$.