1828. *Proposed by Albert F.S. Wong, Temasek Polytechnic, Singapore.*

Evaluate

$$\int_0^1 \left\{\frac{1}{x}\right\}^2 dx,$$

where $\{\alpha\} = \alpha - \lfloor \alpha \rfloor$ denotes the fractional part of α.

Solution by Francisco Vial (student), Pontificia Universidad Católica de Chile, Santiago de Chile. The answer is $\ln(2\pi) - 1 - \gamma$, where γ is the Euler-Mascheroni constant.

Let I be the value of the integral given. For any positive integer k let

$$I_k = \int_{1/(k+1)}^{1/k} \left\{\frac{1}{x}\right\}^2 dx.$$

If $x \in (1/(k+1), 1/k)$, then $1/x \in (k, k+1)$ and therefore $\lfloor \frac{1}{x} \rfloor = k$. With this consideration,

$$I_k = \int_{1/(k+1)}^{1/k} \left(\frac{1}{x} - k\right)^2 dx = 1 + \frac{k}{k+1} - 2k \ln\left(\frac{k+1}{k}\right),$$

by expanding the integrand, followed by direct integration. Now, as $(0,1) = \bigcup_{k \geq 1} I_k$, we have that

$$I = \sum_{k \geq 1} I_k = \sum_{k \geq 1} \left(1 + \frac{k}{k+1} - 2k \ln\left(\frac{k+1}{k}\right)\right).$$

Let (S_n) be the sequence of partial sums. For each $n \in \mathbb{N}$ we have

$$\begin{aligned} S_{n-1} &= \sum_{k=1}^{n-1} \left(1 + \frac{k}{k+1}\right) - 2\sum_{k=1}^{n-1} k \ln\left(\frac{k+1}{k}\right) \\ &= n - 1 + \sum_{k=1}^{n-1} \frac{k}{k+1} - 2\sum_{k=1}^{n-1} k \ln\left(\frac{k+1}{k}\right). \end{aligned}$$

For the first sum we have

$$\sum_{k=1}^{n-1} \frac{k}{k+1} = \sum_{k=1}^{n-1} \left(1 - \frac{1}{k+1}\right) = n - H_n,$$

where H_n is the n–th harmonic number. For the second sum we have

$$\begin{aligned} \sum_{k=1}^{n-1} k \ln\left(\frac{k+1}{k}\right) &= \sum_{k=1}^{n-1} k \Delta \ln k \\ &= n \ln n - 1 \ln 1 - \sum_{k=1}^{n-1} \ln(k+1) \Delta k \\ &= n \ln n - \sum_{k=1}^{n-1} \ln(k+1) = \ln(n^n) - \ln n! = \ln\left(\frac{n^n}{n!}\right), \end{aligned}$$

where Δ is the difference operator (i.e., $\Delta(a_n) = a_{n+1} - a_n$), and we have used summation by parts.

Plugging the two results in the expression for S_{n-1} yields

$$S_{n-1} = 2n - 1 - H_n + 2 \ln\left(\frac{n!}{n^n}\right).$$

According to Stirling's formula,

$$\begin{aligned}\frac{n!}{n^n} &= e^{-n}\sqrt{2\pi n}\left(1+O\left(1/n\right)\right)\\ \ln\left(\frac{n!}{n^n}\right) &= \ln\left(e^{-n}\sqrt{2\pi n}\left(1+O\left(1/n\right)\right)\right)\\ &= -n+\ln\sqrt{2\pi}+\frac{1}{2}\ln(n)+O\left(1/n\right).\end{aligned}$$

Hence,

$$S_{n-1}=\ln(2\pi)-1-H_n+\ln(n)+O(1/n)\to\ln(2\pi)-1-\gamma,$$

which follows from $\gamma=\lim_{n\to\infty}H_n-\ln(n)$.

Proposed: show that

$$\int_0^1\left\{\frac{1}{x^2}\right\}dx=-\zeta\left(\frac{1}{2}\right)-1\approx 0.460354509$$

In fact, for $k\in\mathbb{N}, k\geq 2$, show that

$$\int_0^1\left\{\frac{1}{x^k}\right\}dx=-\zeta\left(\frac{1}{k}\right)-\frac{1}{k-1}.$$

Writing $1/k=z$, with $z\in\mathbb{C}$, $\Re(z)>0$, prove that

$$\zeta(z)=\frac{z}{z-1}+\int_0^1\left\{\frac{1}{x^{1/z}}\right\}dx.$$